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Invariant tangles of coloured non-standard solutions of braid relations

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Abstract. The invariant tangles for Murakami's coloured solution of braid relations are explicitly calculated in terms of the Kauffman–Saleur fermionic integral. A more general coloured solution $R(c_1, c_2)$ of braid group representation is obtained. We verify that such $R(c_1, c_2)$ satisfies all the redundant conditions presented by Murakami. We thus derive invariant Alexander link polynomials for the new coloured solution.

1. Introduction

It is well known that besides the standard solutions of braid relation associated with the Yang–Baxter equation (YBE) there exists a non-standard family that covers a variety of solutions: simple super-extension of standard ones, representation with q at root of unity and a continuous parameter, $Z(N)$ model-type of solutions of BGR (braid group representations), and so on. The corresponding quantum group structures have been discussed in [5–11]. Here we would like to point out that the super-extension is a simple descendant in this family. The simplest one takes the form

$$R = \begin{bmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & 0 & \\ & & & -q^{-1} \end{bmatrix} \tag{1.1}$$

which corresponds to a free fermion model [4, 7, 16] and leads to a quantum group structure with $U(1)$ central element allowed by the quantum double of Drinfeld [15] as shown in [6, 11] or in the super form [7, 17]. Some of the non-standard solutions are connected with Alexandar–Conway link polynomials (ACLP) which are invariant tangles rather than the closure picture of Jones–Kauffman [16–18]. As was discussed in [16, 17] ACLP can be computed with the help of either the state model or the fermionic integral for (1.1) through

$$\begin{aligned} \nabla_K &= q^{-\mathrm{rot}(L)-e(L)} Z_{\mathrm{loop}} \\ &= q^{-\mathrm{rot}(L)-e(L)} \int \mathrm{d}\psi\, \mathrm{d}\psi^{\dagger} \exp(\psi A \psi^{\dagger}) \end{aligned} \tag{1.2}$$

where $e(L)$ is the number of positive crossings minus number of negative crossings, and $\mathrm{rot}(L)$ denotes the number of [↺] minus the number of [↻] in splitting a crossing.

The matrix A in (1.2) comes from

$$\psi A\psi^{\dagger}=\mathscr{A}_{\mathrm{P}}+\mathscr{A}_{\mathrm{I}} \tag{1.3}$$

where the propagator part $\mathscr{A}_{\mathrm{P}}$ and vertex part $\mathscr{A}_{\mathrm{I}}$ are given by

$$\mathscr{A}_{\mathrm{P}}=\sum_{\substack{\text{oriented edge}\\ i,j}}\psi^{\dagger}_{i,\alpha}\psi_{j,\beta}\qquad(\alpha,\beta=u,d) \tag{1.4}$$

$$\mathscr{A}_{\mathrm{I}}=\sum_{\substack{\text{positive}\\ \text{crossing}}}q(\psi_{iu}\psi^{\dagger}_{iu}+\psi_{id}\psi^{\dagger}_{id})+(q^{2}-1)\psi^{\dagger}_{id}\psi_{iu}$$
$$+\sum_{\substack{\text{negative}\\ \text{crossing}}}q^{-1}(\psi_{iu}\psi^{\dagger}_{iu}+\psi_{id}\psi^{\dagger}_{id})+(q^{-2}-1)\psi^{\dagger}_{id}\psi_{iu}. \tag{1.5}$$

Here we denote an over-crossing by u and an under-crossing by d as shown by figure 1.

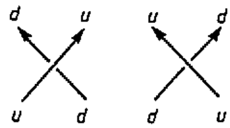


Figure 1.

Following the rule given by Kauffman and Saleur [16, 17], a typical propagator starting from crossing point i and ending at j is illustrated by figure 2.

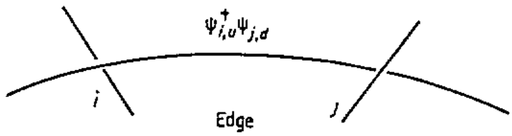


Figure 2.

It is shown that for a diagram in calculating ACLP with (1.1) the fermionic integral formulation is equivalent to the state model in [16]. The result is

$$\nabla_{L}=q^{-\mathrm{rot}(L)-e(L)}\det(A). \tag{1.6}$$

A naturally coloured extension of (1.1) is proposed by Murakami in [19, 20]. The coloured solution has the form

$$R(c_1,c_2)=\begin{bmatrix} t_1 & & & \\ & t_1-t_1^{-1} & t_1t_2^{-1} & \\ & 1 & 0 & \\ & & & -t_2^{-1}\end{bmatrix} \tag{1.7}$$

where t_1 and t_2 are colour-dependent parameters corresponding to colour c_1 and c_2, respectively. Since the structure of the state expansion of (1.7) is the same as that of (1.1) except for the different coefficients, we expect that the fermionic integral computation should give the same result derived by the state model which coincides with the discussion of Murakami on 'redundancy' for (1.7).

In this paper we study ACLP of a non-standard family by a variety of approaches. We first calculate ACLP relating to (1.7) by carrying out the fermionic integral to rederive the result as given in [17]. So far there is no such explicit verification. Next we obtain a new coloured solution of BGR that is more general than (1.7). We obtain all the 'redundant conditions' of Murakami for the new solutions by direct calculations. Hence we immediately establish its invariant tangle picture, i.e. namely its ACLP is well defined and with more parameters.

2. Free fermionic integral for coloured solution of SU(1, 1)

An ACLP is a kind of special link with one strand being open or detached. In order to discuss the coloured case, let us first restrict ourselves to take the non-coloured case into account, which will be useful for later discussions. We put the following proposition:

The fermionic integral (1.2) [16] associated with (1.1) possesses the Reidmeister moves type I, II and III when the following diagrams are connected with any other complicated blocks.

(I) For type I (figures 3 and 4) we have

$$\nabla_{L_1} = \nabla_{L_2}. \tag{2.1}$$

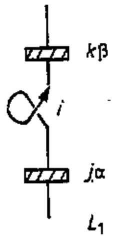


Figure 3.

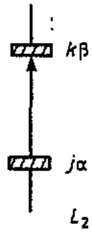


Figure 4.

Proof. For figure 3 we have

$$\mathscr{A}_1 = \mathscr{A}' + \psi^{\dagger}_{j\alpha}\psi_{id} + \psi^{\dagger}_{id}\psi_{iu} + \psi^{\dagger}_{iu}\psi_{k\beta} + q(\psi_{iu}\psi^{\dagger}_{iu} + \psi_{id}\psi^{\dagger}_{id}) + (q^2-1)\psi^{\dagger}_{id}\psi_{iu} \tag{2.2}$$

whereas for figure 4

$$\mathscr{A}_2 = \mathscr{A}' + \psi^{\dagger}_{j\alpha}\psi_{k\beta} \tag{2.3}$$

where $\mathscr{A}'$ stands for the other part connected with $j\alpha$ and $k\beta$ shown in figures 3 and 4 by the hatched blocks. The corresponding matrices A_1 and A_2 in (1.3) are given by†

$$A_1: \quad \begin{array}{c|cccc} & iu & id & \cdots & j\alpha \\ \hline iu & q & -q & & \\ id & & q & & -1 \\ \vdots & & & \multicolumn{2}{c}{\boxed{A_1'}} \\ k\beta & -1 & & & 0 \end{array} \qquad A_2: \quad \begin{array}{c|cc} & \cdots & j\alpha \\ \hline \vdots & \multicolumn{2}{c}{\boxed{A_2'}} \\ k\beta & & -1 \end{array}$$

where the unwritten elements are zero and matrix A_1' differs from A_2' only in the indicated elements (0 in A_1' and -1 in A_2'). By the manipulations that id-row $\times\, q + iu$-row and iu-row $\times\, q + k\beta$-row it is easy to verify that

$$\det A_1 = q^2 \det A_2 \tag{2.4}$$

since under such manipulations the element 0 in the A_1' becomes -1 and hence two matrices A_1' and A_2' become the same:

$$e(L_1) = e(L_2) + 1 \qquad \mathrm{rot}(L_1) = \mathrm{rot}(L_2) + 1. \tag{2.5}$$

† A_1' and A_2' are matrices corresponding to other parts. Note that $\mathscr{A}$ is the same but A_1' and A_2' are no longer the same because of the definition of A (1.4).

Therefore we get

$$\nabla_{L_1} = q^{-\mathrm{rot}(L_1)-e(L_1)} \det A_1 = q^{-\mathrm{rot}(L_2)-e(L_2)} \det A_2 = \nabla_{L_2}.$$

The inverse-oriented diagram possesses the same property.

(II) For the unitarity of type II (figures 5 and 6) we have

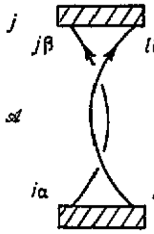


Figure 5.

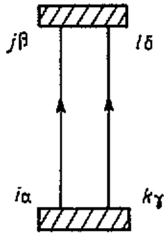


Figure 6.

$$\mathscr{A}_1 = \mathscr{A} + \psi^\dagger_{md}\psi_{j\beta} + \psi^\dagger_{mn}\psi_{l\delta} + \psi^\dagger_{i\alpha}\psi_{nd} + \psi^\dagger_{k\gamma}\psi_{nu} + (q^2-1)\psi^\dagger_{md}\psi_{mu} + (q^{-2}-1)\psi^\dagger_{nd}\psi_{nu}$$
$$+ q(\psi_{mu}\psi^\dagger_{mu} + \psi_{md}\psi^\dagger_{md}) + q^{-1}(\psi_{nu}\psi^\dagger_{mu} + \psi_{nd}\psi^\dagger_{nd}) + \psi^\dagger_{nu}\psi_{mu} + \psi^\dagger_{nd}\psi_{md} \quad (2.6)$$

$$\mathscr{A}_2 = \mathscr{A} + \psi^\dagger_{i\alpha}\psi_{j\beta} + \psi^\dagger_{k\gamma}\psi_{l\delta} \quad (2.7)$$

where $\mathscr{A}$ is the rest part connected with $j\beta$, $l\delta$ and $i\alpha$, $k\gamma$ whose matrices in (1.3) are A''_1 and A''_2 with differences in two elements as shown in the blocks of A_1 and A_2:

A_1:

	mu	md	nu	nd	$\dots$	$i\alpha$	$\dots$	$k\gamma$
mu	q	$1-q$	-1					
md		q		-1				
nu			q	$1-q$				-1
nd				q		-1		
$\vdots$							A''_1	
$j\beta$		-1				0		
$\vdots$								
$l\delta$	-1							0

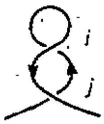
A_2:

	$\dots$	$i\alpha$	$\dots$	$k\gamma$
$\vdots$			A''_2	
$j\beta$		-1		
$\vdots$				
$l\delta$				-1

By making a series of manipulations such as md-row $\times\, q + j\beta$-row, ... the elements 0 in the matrix A''_1 become -1 which is the same as A''_2 in A_2 and the relation is found

$$\det A_1 = \det A_2$$
$$e(L_1) = e(L_2) + 1 - 1 = e(L_2) \quad (2.8)$$
$$\mathrm{rot}(L_1) = \mathrm{rot}(L_2).$$

For the tangled cross-channel unitarity of type II we have (for simplicity the rest part is omitted)



$$\mathscr{A}=\psi^\dagger_{jd}\psi_{id}+\psi^\dagger_{id}\psi_{iu}+\psi^\dagger_{iu}\psi_{ju}$$
$$+q^{-1}(\psi_{iu}\psi^\dagger_{iu}+\psi_{id}\psi^\dagger_{id})+(q^{-2}-1)\psi^\dagger_{id}\psi_{iu}$$
$$+q(\psi_{ju}\psi^\dagger_{ju}+\psi_{jd}\psi^\dagger_{jd})+(q^{2}-1)\psi^\dagger_{j\alpha}\psi_{ju}$$

	iu	id	ju	jd
iu	q^{-1}	$1-q^{-2}$	-1	
id		q^{-1}		-1
ju	-1		q	$1-q^2$
jd				q

$$\det A=1 \qquad e(L)=0,\ \mathrm{rot}(L)=0 \qquad \nabla_L=1=\nabla \curlywedge. \tag{2.9}$$

(III) For type III (figures 7 and 8) correspondingly we have:

$$\mathscr{A}_1=\mathscr{A}+\psi^\dagger_{i\alpha}\psi_{iu}+\psi^\dagger_{iu}\psi_{ju}+\psi^\dagger_{ju}\psi_{i'\alpha'}+\psi^\dagger_{m\beta}\psi_{ku}+\psi^\dagger_{ku}\psi_{jd}+\psi^\dagger_{jd}\psi_{m'\beta'}$$
$$+\psi^\dagger_{n\gamma}\psi_{kd}+\psi^\dagger_{kd}\psi_{id}+\psi^\dagger_{id}\psi_{n'\gamma'}+q(\psi_{iu}\psi^\dagger_{iu}+\psi_{id}\psi^\dagger_{id})$$
$$+(q^2-1)\psi^\dagger_{id}\psi_{iu}+q(\psi_{ju}\psi^\dagger_{ju}+\psi_{jd}\psi^\dagger_{jd})+(q^2-1)\psi^\dagger_{jd}\psi_{ju}$$
$$+q(\psi_{ku}\psi^\dagger_{ku}+\psi_{kd}\psi^\dagger_{kd})+(q^2-1)\psi^\dagger_{kd}\psi_{ku}$$

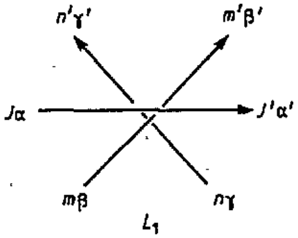


Figure 7.

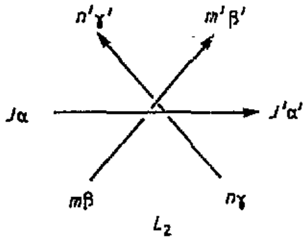


Figure 8.

and

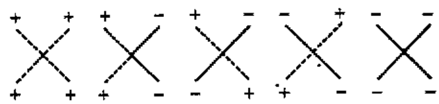
$$\mathscr{A}_2=\mathscr{A}+\psi^\dagger_{i\alpha}\psi_{iu}+\psi^\dagger_{iu}\psi_{ju}+\psi^\dagger_{ju}\psi_{i'\alpha'}+\psi^\dagger_{m\beta}\psi_{id}+\psi^\dagger_{id}\psi_{ku}+\psi^\dagger_{ku}\psi_{m'\beta'}$$
$$+\psi^\dagger_{n\gamma}\psi_{jd}+\psi^\dagger_{jd}\psi_{kd}+\psi^\dagger_{kd}\psi_{n'\gamma'}+q(\psi_{iu}\psi^\dagger_{iu}+\psi_{id}\psi^\dagger_{id})+(q^2-1)\psi^\dagger_{id}\psi_{iu}$$
$$+q(\psi_{ju}\psi^\dagger_{ju}+\psi_{jd}\psi^\dagger_{jd})+(q^2-1)\psi^\dagger_{jd}\psi_{ju}+q(\psi_{ku}\psi^\dagger_{ku}+\psi_{kd}\psi^\dagger_{kd})$$
$$+(q^2-1)\psi^\dagger_{kd}\psi_{ku}.$$

The corresponding matrices A_1 and A_2 can be tabled. Omitting the details we obtain

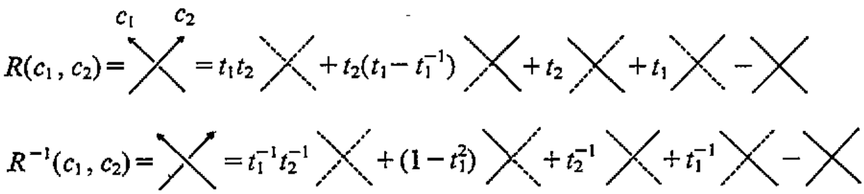
$$\det A_1=\det A_2,\ e(L_1)=e(L_2)$$
$$\mathrm{rot}(L_1)=\mathrm{rot}(L_2) \tag{2.9}$$
$$\nabla_{L_1}=\nabla_{L_2}.$$

Therefore the proposition is proved. □

It is worth noting that the above properties are guaranteed by the structure of state expansions and do not depend on the coefficients before in the state expansions if the braid relation is satisfied. This is the point of the consistency between the state model and fermionic integral. Moreover, the coloured solution (1.7) possesses the same state decomposition as for (1.1). The difference is only in the coefficients. If we follow [16, 21] by using a dashed line to denote the boson (spin +) and a solid line for the fermion (spin −), i.e.



then for (1.7) we have the state expansion



where $t_i = t_{c_i}$ $(i = 1, 2)$ are colour dependent parameters. Based on the state expansion we immediately know that the fermionic integral formulation is still valid for the solution (1.7). Keeping in mind

$$\nabla_B = \prod_{i=1}^{N} t_i Z^c_{\text{loop}} \qquad (i\text{: colour index}) \tag{2.10}$$

where

$$Z^c_{\text{loop}} = \sum_{\Lambda}^{N} (-1)^{\text{fermion loop}} \prod_{\text{crossing}} \text{Boltzmann weights}$$

and Λ is the loop configuration, open last strand, carrying a bosonic or fermionic state, and N is the number of strings of coloured braid corresponding to a certain line.

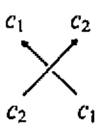
The coloured fermionic integral is given by

$$Z^c_{\text{loop}} \sim \int \mathrm{d}\psi \, \mathrm{d}\psi^\dagger \exp(\mathscr{A}(c)) \tag{2.11}$$

where ψ and $\psi^\dagger$ have the same meaning as in the non-coloured case. Again

$$\mathscr{A}(c) = \mathscr{A}_P + \mathscr{A}_I(c)$$

where the propagator part $\mathscr{A}_P$ is the same as the non-coloured one shown by (1.4). The major difference is in the interaction part $\mathscr{A}_I(c)$ because for the coloured case we have to distinguish between down-colour and up-colour parameters; for example



c_1: down colour

c_2: up colour.

To simplify notation one denotes by

$$t_{c_i}^{(u)} = t_i^{(u)} \qquad t_{c_i}^{(d)} = t_i^{(d)}.$$

In comparison to the non-coloured case we factor out

$$\prod_{\substack{\text{positive}\\ \text{crossing}}} (t_m^{(u)})^{-1} \prod_{\substack{\text{negative}\\ \text{crossing}}} (t_n^{(d)}).$$

By parallelizing the same arguments of Kauffman–Saleur, the coloured counterpart $\mathscr{A}_{\mathrm{I}}(c)$ of $\mathscr{A}_{\mathrm{I}}$ takes the form

$$\mathscr{A}_{\mathrm{I}}(c) = \sum_{\substack{\text{positive}\\ \text{crossing}}} \{t_m^{(d)}\psi_{id}\psi_{id}^{\dagger} + t_n^{(u)}\psi_{iu}\psi_{iu}^{\dagger} + t_n^{(u)}(t_m^{(d)} - t_m^{(d)^{-1}})\psi_{id}^{\dagger}\psi_{iu}\}$$
$$+ \sum_{\substack{\text{negative}\\ \text{crossing}}} \{t_m^{(u)^{-1}}\psi_{id}\psi_{id}^{\dagger} + t_n^{(d)^{-1}}\psi_{iu}\psi_{iu}^{\dagger} + t_n^{(u)^{-1}}(t_m^{(d)^{-1}} - t_m^{(d)})\psi_{id}^{\dagger}\psi_{iu}\}.$$

We thus have

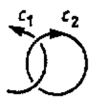
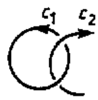
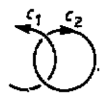
$$\mathscr{A}(c) = \mathscr{A}_{\mathrm{P}} + \mathscr{A}_{\mathrm{I}}(c) = \psi_{i\alpha}A^c_{i\alpha,j\beta}\psi_{j\beta}^{\dagger}$$

where A^c is a $2n\times 2n$ matrix where n is the number of crossing points and

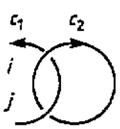
$$Z^c_{\text{loop}} = \left\{ \prod_{\substack{\text{positive}\\ \text{crossing}}} (t_m^{(u)})^{-1} \prod_{\substack{\text{negative}\\ \text{crossing}}} t_n^{(d)} \right\} \det(A(c)). \tag{2.12}$$

To emphasize the consistency between the state model and the fermionic integral formulation explicitly in the coloured case we list some results for Z^c_{loop} calculated by the state model and the path integral independently

Z^c_{loop}

coloured link				
state model	$t_1^{-1}(t_2 - t_2^{-1})$	$t_2(t_1 - t_1^{-1})$	$t_2^{-1}(t_1^{-1} - t_1)$	$t_1(t_2^{-1} - t_2)$
path integral	same			

The results are the same as those given by Murakami [19, 20]. For instance for:



$$\mathscr{A} = \psi_{iu}^{\dagger}\psi_{jd} + \psi_{jd}^{\dagger}\psi_{iu} + \psi_{ju}^{\dagger}\psi_{id} + t_1\psi_{id}\psi_{id}^{\dagger} + t_2\psi_{iu}\psi_{iu}^{\dagger}$$
$$+ (t_1 - t_1^{-1})t_2\psi_{id}^{\dagger}\psi_{iu} + t_2\psi_{jd}\psi_{jd}^{\dagger} + t_c\psi_{ju}\psi_{ju}^{\dagger}$$
$$+ t_1(t_2 - t_2^{-1})\psi_{jd}^{\dagger}\psi_{ju}.$$

The corresponding matrix $A(c) = A^c$ is:

	iu	id	ju	jd
iu	t_2	$t_2(t_1^{-1} - t_1)$		-1
id		t_1	-1	
ju			t_1	$t_1(t_2^{-1} - t_2)$
jd	-1			t_2

Hence we obtain

$$\det(A(c)) = t_2(t_2 - t_2^{-1})$$
$$Z^c_{\text{loop}} = t_2^{-1}(t_2 - t_2^{-1}).$$

To conclude this section we would like to point out that for the simple coloured solution (1.7) the coloured state gives the same tangle picture in accordance with the discussion on the redundancy in [19, 20] based on the Markov trace. Because the solution (1.7) is not unique a new family of solutions of

$$R_{12}(\lambda, \mu)R_{23}(\lambda, \nu)R_{12}(\mu, \nu) = R_{23}(\mu, \nu)R_{12}(\lambda, \nu)R_{23}(\lambda, \mu) \tag{2.13}$$

can be derived [22]. It is natural to set up ACLP for such new solutions. In the next section we first obtain more a general solution with colours and then establish their invariant tangle through proving the redundant conditions proposed by Murakami [19, 20].

3. New non-standard solutions of the coloured braid group

To extend solution (1.7) we substitute into (2.13) the following matrix form

$$R(\lambda, \mu) = \sum_a u_a(\lambda, \mu)E_{aa} \otimes E_{aa} + w(\lambda, \mu)E_{-\frac{1}{2}\,-\frac{1}{2}} \otimes E_{+\frac{1}{2}\,+\frac{1}{2}} + \sum_{a \neq b} p^{(a,b)}(\lambda, \mu)E_{ab} \otimes E_{ba} \tag{3.1}$$

where a, b may be $+\frac{1}{2}$ or $-\frac{1}{2}$, and λ and μ are colour parameters. Unknown colour-dependent parameters $u_a(\lambda, \mu)$, $p^{(a,b)}(\lambda, \mu)$ and $W(\lambda, \mu)$ are to be determined by substituting (3.1) into (2.13).

After calculation we derive the following solution

$$R(\lambda, \mu) = f(\lambda, \mu)\begin{bmatrix} q & & & \\ & 0 & \eta t^\alpha_\lambda & \\ & \eta^{-1} t^\beta_\mu & \tilde{w}(\lambda, \mu) & \\ & & & -q^{-1}t^\alpha_\lambda t^\beta_\mu \end{bmatrix} \tag{3.2}$$

where

$$\begin{aligned} & t^\alpha_\lambda = Q^{\Sigma^m_{k=1}\tilde{\alpha}_k\lambda^k} \qquad t^\beta_\mu = Q^{\Sigma^m_{k=1}\tilde{\beta}_k\mu^k} \\ & \tilde{\alpha}_k = \alpha_k(-\tfrac{1}{2}) - \alpha_k(\tfrac{1}{2}) \\ & \tilde{\beta}_k = \beta_k(-\tfrac{1}{2}) - \beta_k(\tfrac{1}{2}) \\ & f(\lambda, \mu) = Q^{\Sigma^m_{k=1}\{\alpha k(\frac{1}{2})\lambda^k + \beta k(\frac{1}{2})\mu^k\}} \end{aligned} \tag{3.3}$$

and $\tilde{w}(\lambda, \mu)$ satisfies

$$\tilde{w}(\lambda, \mu)\tilde{w}(\mu, \nu) = \{q - q^{-1}t^\alpha_\mu t^\beta_\mu\}\tilde{w}(\lambda, \nu). \tag{3.4}$$

The details of the calculations can be found in [22]. It can also be checked directly. By defining

$$\tilde{w}(\lambda, \mu) = t^\beta_\mu \bar{W}(\lambda, \mu) \qquad q(t^\beta_\mu)^{-1} = s_\mu$$
$$q^{-1}t^\alpha_\lambda = t_\lambda \qquad q(t^\alpha_\mu)^{-1} = t_\mu^{-1} \qquad q(t^\beta_\lambda) = s_\lambda^{-1}$$

and dispensing with the trivial factor t_μ^β we get

$$R(\lambda,\mu)=\begin{bmatrix} s_\mu & & & \\ & 0 & \eta t_\lambda s_\mu & \\ & \eta^{-1} & \bar{W}(\lambda,\mu) & \\ & & & -t_\lambda \end{bmatrix} \tag{3.5}$$

$$\bar{W}(\lambda,\mu)\bar{W}(\mu,\nu)=(t_\mu-s_\mu)\bar{W}(\lambda,\nu). \tag{3.6}$$

A particular solution of (3.6) is

$$\bar{W}(\lambda,\mu)=(t_\lambda)^{-1}t_\mu(t_\mu-s_\mu). \tag{3.7}$$

If $R(\lambda,\mu)$ is a solution, so is $R(\mu,\lambda)$. Therefore $(R(\mu,\lambda))^{-1}$ should be a solution of (2.13), namely,

$$R(c_1,c_2)=\begin{bmatrix} t_1 & & & \\ & t_1t_2^{-1}(t_1-s_1) & t_1s_2 & \\ & 1 & 0 & \\ & & & -s_2 \end{bmatrix} \tag{3.8}$$

where we have taken $\eta=1$. It can be checked out that (3.8) is really a solution of (2.13). In (3.8) colour-dependent parameters t_1, t_2, s_1 and s_2 are free parameters instead of t_λ, s_μ for later convenience in a discussion of the n-colour case. Obviously this is a more general solution than (1.7). When $s=t_1^{-1}$ and $s=t_2^{-1}$, it returns to the solution of [19].

4. ACLP for extended coloured solution

Following the general arguments of the Markov trace [19, 23], in order to establish ACLP associated with our new solution, we first present an enhanced YB operator in our case. For (3.8) we find that the necessary entries h, $R(c_1,c_2)$, $\alpha(c)$ and $\beta(c)$ satisfy the following properties:

(a) $$h=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{4.1}$$

(b) $$R(c_1,c_2)(h\otimes h)=(h\otimes h)R(c_1,c_2) \tag{4.2}$$

(c) $$\mathrm{Tr}_2(R(c,c)(I\otimes h))=\alpha(c)\beta(c) \tag{4.3}$$

$$\mathrm{Tr}_2(R^{-1}(c,c)(I\otimes h))=\alpha^{-1}(c)\beta(c) \tag{4.4}$$

where Tr_2 means that trace is taken in the second space only and leaves the first space free. The multivariable polynomial for a coloured link b is then given by

$$\alpha(c)=(t_1s_1)^{1/2} \qquad \beta(c)=(s_1t_1^{-1})^{1/2} \tag{4.5}$$

$$\nabla(b)=\prod_{c=1}^{\infty}(\alpha(c))^{-W_{(b)}^{(c)}}\left(\prod_{k=1}^{n}\beta(c_k)^{-1}\right)\mathrm{Tr}\{B\otimes(h\otimes h\otimes\ldots\otimes h)\} \tag{4.6}$$

where $W^{(c)}(b)$ denotes the number of crossings in b such that the strings of over-path and under-path are both coloured by c. B is the closed braid form of b and composed of an $R(c_i,c_j)$ operator due to the Alexander theorem. Unfortunately the general formula (4.6) leads to a trivial result for any links since $\mathrm{tr}(h)=0$. To avoid this triviality we should discuss the redundant conditions needed for constructing ACLP, namely we should look for the sufficient conditions for the existence of invariant tangle associated with solution (3.8).

First we introduce several notations:

$B_n^{c_1c_2\cdots c_n}$ is the coloured braid group formed by n coloured strings separated by colours $c_1, c_2, \ldots, c_n$ in which some of the colours may happen to be the same.

$\mathrm{Tr}_n(B)$ $(\forall B\in B_n^{c_1c_2\cdots c_n})$ means that we take trace of B in the nth (right-most) space only where B is defined on $V^{c_1}\otimes V^{c_2}\otimes\ldots\otimes V^{c_n}$.

$$\mathrm{Tr}_{n,i}(B)=\mathrm{Tr}_i(\mathrm{Tr}_{i+1},\ldots(\mathrm{Tr}_n(B)\ldots)\ldots)\qquad (i<n). \tag{4.7}$$

Note that

$$\begin{aligned}\nabla(b)&\sim\mathrm{Tr}(B(h\otimes h\ldots\otimes h))\\&=\mathrm{Tr}_{n,1}(B(h\otimes h\ldots\otimes h))\\&=\mathrm{Tr}_1(h\,\mathrm{Tr}_{n,2}(B\cdot(I\otimes h\otimes\ldots\otimes h)))\end{aligned} \tag{4.8}$$

so that if $\mathrm{Tr}_{n,2}(B(I\otimes h\ldots\otimes h))$ is a scalar matrix (number times the unity matrix) then the triviality occurs for Tr_1 (h scalar)$=0$ $\forall B\in B_n^{c_1\cdots c_n}$ and can then be separated by taking out the redundant trace $\mathrm{Tr}_1(h)$ itself. Such $R(c_1, c_2)$ is called redundant [19, 20]. If $R(c_1, c_2)$ is redundant then the scalar of $\mathrm{Tr}(B(I\otimes h\otimes\ldots\otimes h))$ is invariant of link to some factor.

In this section we shall prove that our general solution (3.8) is redundant.

First we introduce some further notation. Define

$$\begin{aligned}r(c_1, c_2)&=R(c_1, c_2)R(c_2, c_1)\\r_1(c_1, c_2)&=r(c_1, c_2)\otimes I\\r_2(c_1, c_2)&=I\otimes r(c_1, c_2).\end{aligned} \tag{4.9}$$

Note that $r(c_1, c_2)$ thus defined has two eigenvalues

$$r(c_1, c_2)+r^{-1}(c_1, c_2)=(t_1t_2+s_1s_2) \tag{4.10}$$

and taking the above definition into account by parallelizing [19] we derive the following results concerning the redundant conditions:

(1) 1-string case:
$B_1^{c_1}$ is generated by identity I.

(2) 2-string case:
$B_2^{c_1c_2}$ is generated by I, $r(c_1, c_2)$ due to (4.10).
$\mathrm{Tr}_2\, r(c_1, c_2)=$scalar I and I, $r(c_1, c_2)$ also serve the basis of $B_2^{c_1c_2}$.

(3) 3-string case:
(a) I, r_1, r_2, r_1r_2, r_2r_1, and $r_1r_2r_1$ serve the basis of $B_3^{c_1c_2c_3}$. They are independent.

(b) Other combinations of r_1 and r_2, for example, $r_2r_1r_2$. After lengthy calculations we have

$$(t_1t_2-s_1s_2)r_2r_1r_2-(t_2t_3-s_2s_3)r_1r_2r_1$$
$$=t_2s_2(t_1s_3-t_3s_1)(r_1r_2+r_2r_1)$$
$$+t_2s_2(t_1s_2t_3s_3-t_1s_2s_3^2-t_1t_2t_3^2+s_1s_2s_3^2-t_2t_3s_1s_3+t_2t_3^2s_1)r_1$$
$$+t_2s_2(t_1^2t_2t_3-t_1^2t_2s_3+t_1t_2s_1s_3-t_1t_3s_1s_2-s_1^2s_2s_3+t_3s_1^2s_2)r_2$$
$$+t_2^2s_2^2(t_1-s_1)(t_3-s_3)(s_1t_3-s_3t_1).$$

In the following we explicitly verify the statement (a).

(c) $c=c_1=c_2=c_3$ (non-coloured case).

B is generated by I, $R(c,c)\otimes I$, $I\otimes R(c,c)$

$$R^2(c,c)=(t-s)R(c,c)+ts \tag{4.11}$$

By recalling $R(c,c)\to R(c,c)/\sqrt{ts}$, $q=\sqrt{t/s}$ which is nothing but the skein relation of Jones type.

(d) $c_1=c_2\neq c_3$ (and $c_1\neq c_2=c_3$ is similar).

$B_n^{c_1c_2c_3}$ is generated by I, $R(c_1,c_1)\otimes I$, $I\otimes r(c_1,c_3)=r_2R(c_1,c_1)=(t_1-s_1)^{-1}R^2(c_1,c_1)-t_1s_1(t_1-s_1)^{-1}$.

(e) $c_1=c_3\neq c_2$.

$B_3^{c_1c_2c_3}$ is generated by I, r_1, r_2 and $(R(c_1,c_2)\otimes I)(I\otimes R(c_1,c_1)((R(c_1,c_2)^{-1}\otimes I)$, but but we have $(\Delta=t_1^2t_2-t_1s_1s_2+s_1^2s_3)$:

$$(R(c_1,c_2)\otimes I)(I\otimes R(c_1,c_1)((R(c_1,c_2)^{-1}\otimes I)$$
$$=\frac{t_1s_1(t_2-s_2)}{s_1s_2-t_1t_2}+\frac{t_1s_2-t_2s_1}{(s_2-t_2)(t_2t_1-s_1s_2)}\cdot r_1$$
$$+\frac{t_1^2t_2+s_1^2s_2}{(t_2-s_2)\Delta}r_2+\frac{t_1t_2^2+s_1s_2^2}{t_2s_2(s_2-t_2)\Delta}r_1r_2$$
$$+\frac{t_1+s_1}{(s_2-t_2)\Delta}r_2r_1-\frac{s_2+t_2}{s_2t_2(s_2-t_2)\Delta}r_1r_2r_1. \tag{4.12}$$

(f) $c_1\neq c_2$, $c_2\neq c_3$ and $c_1\neq c_3$.

$B_3^{c_1c_2c_3}$ is generated by I, r_1, r_2, and

$$(R(c_1,c_2)\otimes I)(I\otimes R(c_1,c_3))(I\otimes R(c_3,c_1))(R^{-1}(c_1,c_2)\otimes I)$$

however, which is equal to

$$\{(t_1s_2-t_2s_1)(t_1t_2s_3-s_1s_2t_3)+(t_3-s_3)(t_1s_2-t_2s_1)r_1$$
$$+(s_1^2s_2-t_1^2t_2)r_2+((t_1t_2^2+s_1s_2^2)/s_2t_2)r_1r_2$$
$$+(s_1+t_1)r_2r_1-(s_2+t_2)r_1r_2r_1\}/\Delta' \tag{4.13}$$

where

$$\Delta' = (t_1 t_2 - s_1 s_2)(s_2 - t_2).$$

Summing up the above calculations we have proved that the bases of $B_3^{c_1 c_2 c_3}$ are I, r_1, r_2, $r_1 r_2$, $r_2 r_1$, $r_1 r_2 r_1$.

It is worth noting that in such six-basis a non-trivial operation defined on third (right-most) space V^{c_3} is r_2 which occurs at most once, say, zero time for I, r_1, one time for $r_1 r_2, r_2, r_1 r_1, r_1 r_2 r_1$. Taking the fact that $\mathrm{Tr}_2(r(c_1, c_1)(I \otimes h)) = \text{scalar}$ into account we obtain

$$\mathrm{Tr}_3(B(I \otimes I \otimes h)) \in B_2^{c_1 c_2} \qquad \forall B \in B_3^{c_1 c_2 c_3}. \tag{4.14}$$

(g) A similar procedure can be applied to the n-string case in terms of reduction. It has been proved in [19, 20] that

$$\mathrm{Tr}_n(B(I \otimes I \otimes \ldots \otimes h)) \in B_{n-1}^{c_1 \cdots c_{n-1}} \qquad \forall B \in B_n^{c_1 \cdots c_n} \tag{4.15}$$

namely, three neighbouring strings generate the general properties of n-strings for a Yang–Baxter system.

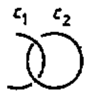
(h) Summing up the above discussions we conclude that

$$\mathrm{Tr}_{n,2}(B(I \otimes h \otimes \ldots \otimes h)) \in B_1^{c_1} \tag{4.16}$$

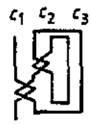
which is a scalar, namely $R(c_1, c_2)$ is redundant. We then can leave one string to be opened and close other strings to form an invariant tangle. The final form of the invariant is given by

$$\nabla'(b) = (t_1 - s_1)^{-1} \left(\prod_{c=1}^{\infty} (\alpha(c))^{-W^{(c)}(b)} \right) \left(\prod_{k=1}^{n} \beta_k^{-1} \right) \mathrm{Tr}_{n,2}(B(I \otimes h \otimes \ldots \otimes h)). \tag{4.17}$$

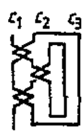
For illustration we give some examples



$$\nabla' = (t_1 t_2 s_2 / s_1)^{1/2} \tag{4.18}$$



$$\nabla' = (t_1 t_2 t_3 s_2 s_3 / s_1)^{1/2} (t_2 - s_2) \tag{4.19}$$



$$\nabla' = (t_1 t_2 t_3 s_2 s_3 / s_2)(t_2 - s_2)(t_1 t_2 + s_1 s_2). \tag{4.20}$$

In the above examples the subindices 1, 2 and 3 of parameters t and s correspond to colours c_1, c_2 and c_3, respectively.

5. Conclusions

It is well known that for standard solutions of the braid relation within the six-vertex model, the state model of Kauffman is universal in constructing link polynomials equivalent to the Markov trace approach. For some non-standard solutions the invariant tangle should be studied. Obviously not all complicated tangles with incoming index a and outgoing index b turn out to be δ_a^b scalar. It does occur in the case where taking trace in the first (left-most) space is redundant. Of course, redundant conditions are stringent and model dependent. For some models the state model is established for invariant tangles and turns out to be equivalent to the 'redundancy' picture.

In general the proof of satisfaction of redundancy is lengthy and strongly model dependent. The translation of the language of Markov trace to the state model is also model dependent. From a practical point of view, we could say that the invariant tangle (ACLP) theory is still at the beginning.

Before ending this section we would like to make some remarks.

(1) Not all super-solutions of BGR should receive a tangle picture. For instance, for non-standard solutions associated with $B(n)$ the link polynomials are still 'standard'. Only for $C(n)$ and $D(n)$ should it be dealt with as an invariant tangle. Such solutions come from the reducibility of the Birman–Wenzl algebra. This problem has been solved since we can prove that any R-matrix satisfying BW algebra is always redundant.

(2) Another example is the non-standard solutions associated with spin model recently discussed in [25]. We can prove that the 9×9 solution is definitely not a BW algebra but is redundant [26].

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